On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

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Overview

minimal resolution		Kleinian singularity	
$\widetilde{X}$	$\xrightarrow{\pi}$	$X \circlearrowleft \theta$	anti-Poisson involution
$\cup$		$\cup$	
$\pi^{-1}(X^\theta)$		X^θ	Lagrangian subvariety

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes.

Kleinian singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup.
The **Kleinian singularity** is the quotient $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u, v]^\Gamma$.

Example

$\Gamma = \{\pm I_2\}$,
$\mathbb{C}[u, v]^\Gamma = \mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x, y, z]/(xy - z^2)$.

Fact (Klein, 1884)

$\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ has an isolated singularity at 0.

Minimal resolutions

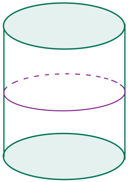
McKay correspondence: Kleinian singularities are in bijection with ADE Dynkin diagrams.

$\pi\colon \widetilde{X} \to X$, minimal resolution.

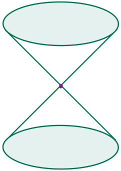
$\pi^{-1}(0)$ = union of $\mathbb{P}^1$'s, according dually to ADE Dynkin diagrams.

Examples:

$T^*\mathbb{P}^1$

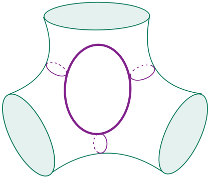


$\xrightarrow{\pi}$

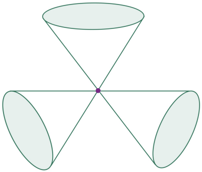


A_1 singularity

$xy - z^2 = 0$



$\xrightarrow{\pi}$



D_4 singularity

$x^3 + xy^2 + z^2 = 0$

Anti-Poisson involutions & fixed point loci

Set $X := \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u, v]^\Gamma$ is a graded Poisson algebra with **Poisson bracket**

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Definition: An **anti-Poisson involution** of $X = \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\},\ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Definition: The **fixed point locus** is $X^\theta := \operatorname{Spec}\mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$.

Anti-Poisson involution & fixed-point loci

Proposition 1 (H.)

There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by graded Poisson automorphisms.

Proposition 2 (H.)

- The fixed point locus X^θ is reduced.
- If X^θ is not a single point, each irreducible component of X^θ is either $\mathbb{A}^1$ or a cusp.

Example: Type A_n singularity $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$. Then θ swapping $x \leftrightarrow y$ is an anti-Poisson involution. We have

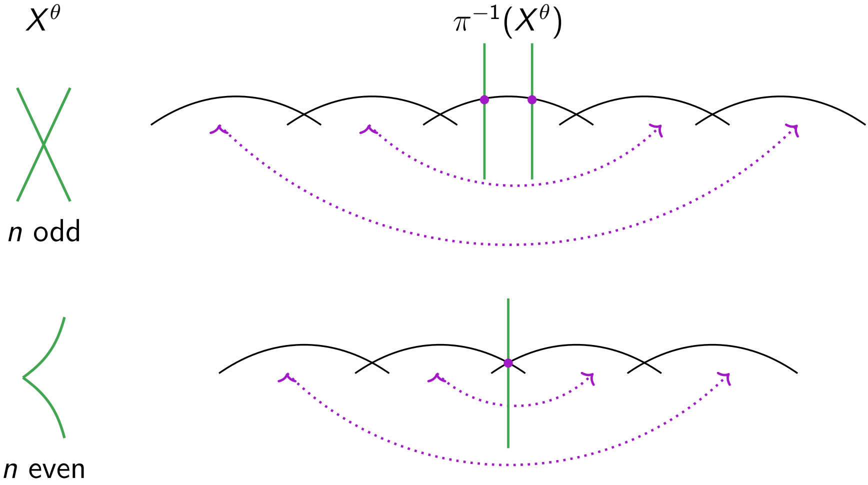
$$X^\theta = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x, z]/(x^2 - z^{n+1}),$$

which is a union of two $\mathbb{A}^1$'s when n is odd, a cusp when n is even.

Preimage of fixed point loci under minimal resolutions

Recall $\pi\colon \widetilde{X} \to X$ minimal resolution. $0 \in X^\theta \Rightarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$.

Example: Consider type A_n singularities with θ swapping $x \leftrightarrow y$.



Lift anti-Poisson involutions

$\pi : \widetilde{X} \to X$, minimal resolution, and θ an anti-Poisson involution of X.

Theorem (H.)

There exists a unique anti-symplectic involution $\tilde{\theta} : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$. It can be constructed explicitly via quiver varieties.

We have $\pi^{-1}(X^\theta) = \pi^{-1}(0) \cup \widetilde{X}^{\tilde{\theta}}$.

Fact: $\widetilde{X}$ smooth $\Rightarrow$ $\widetilde{X}^{\tilde{\theta}}$ is smooth Lagrangian (no intersection, no cusp).

Remark: The preimage $\pi^{-1}(X^\theta)$ is **NOT** reduced in general.

Write $\pi^{-1}(X^\theta) = \sum_{j=1}^m 1 \cdot L_j + \sum_{i=1}^n a_i C_i$ as a divisor, where $L_j \simeq \mathbb{A}^1$, $C_i \simeq \mathbb{P}^1$. There are methods to determine the multiplicities a_i.

Thank You!